

Geometric flows

Postdoc seminar
SLGP 1/21/15.

1

Heat equations for various geometric objects.

Heat $\sim$ curvature (try to smooth out & arrive at sth optimal.)

~~Parabolic approach to solving elliptic equations.~~

(Field = ^{old} Examples + ^{elementary} techniques) only way to learn the techniques is through examples!

(1) Harmonic map flows. $u: M \rightarrow N \subset \mathbb{R}^K$ Riemann manifolds

$$\partial_t u = (\Delta u)^T \xrightarrow{t \rightarrow \infty} (\Delta u)^T = 0$$



$M = D^2$



$N = S^2$

Harmonic map.

(Nonlinear σ -model)

i.e.: critical pt. of Dirichlet energy

$$D(u) = \int_M |\nabla u|^2 dV.$$

E.g. assume $D^2 \rightarrow S^2$ is rotationally symmetric:



$$\partial_t \theta(r, t) = \underbrace{\partial_{rr} \theta}_{\text{diffusion}} + \underbrace{\frac{1}{r} \partial_r \theta}_{\text{reaction (nonlinearity)}} - \frac{1}{2r^2} \sin 2\theta.$$

diffusion

reaction (nonlinearity)

Can this be solved for all $t \in [0, \infty)$? No. (1992).

But assume N has negative sectional curvature.

e.g. $N = H^k$

Theorem (Eells-Sampson, 1964). HMF exists for

(first fairly

flow theorem)

all time and subconverges to a harmonic map.

Cor.: Every map $M \xrightarrow{\text{(both compact)}} N$ is homotopic to

a harmonic map. $(u_0 \rightsquigarrow u(t) \rightsquigarrow u_\infty)$

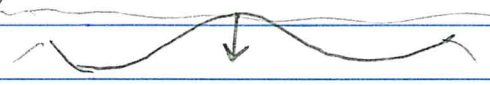
Proof of E-S: $e(u) = |\nabla u|^2$, Bochner f'la: ≤ 0

$$(\partial_t - \Delta)e(u) = -|\nabla du|^2 - \langle Ric^M_{ij} \nabla_i u, \nabla_j u \rangle + \langle Rm^N(\nabla_i u, \nabla_j u) \nabla_i u, \nabla_j u \rangle$$

(heat kernel diffusion)

$$(\partial_t - \Delta)e(u) \leq C \cdot e(u).$$

$$(\partial_t - \Delta)(e^{-C(t-t_0)} e(u)) \leq 0 \quad (*)$$

max. prin: 

$$\sup e(u(t)) \leq e^{-C(t-t_0)} \sup e(u(t_0)).$$

$$\Rightarrow |\nabla u| < \infty \text{ at finite time.}$$

Very-time exist.

Convergence:

MF is grad-flow of $D(u) = \int e(u)$.

$$D(u(t)) \searrow$$

(monotonicity)
Moser's Harnack inequality: applied to (*):

$$e(u(t)) \leq C \int_{t-1}^t \int_M e(u) \leq C D(u(t)) \leq C D(u_0)$$

$$\Rightarrow u(t) \text{ is } C' \text{-bdd.}$$

Ascoli-Arscoli: $u(t_i) \rightarrow u_\infty$ for some $t_i \rightarrow \infty$

② Hamilton's Ricci flow. (1st-order d.t. flow?)

Evolution eqn for Riemannian metric

$$\partial_t g_{ij} = -2 R_{ij}.$$

Parabolic Einstein equation, $\left(\begin{array}{l} \text{Hyp. Es eq: physics} \\ \text{Ell. Es eq: geometry} \end{array} \right)$

Thm (1982). 3-manifolds with positive

Ricci curvature are space-forms (S^3/Γ).

Evolution of full curvature (Bachman '16):

$$(\partial_t - \Delta_{g(t)}) R_m = Q(R_m).$$

diffusion.

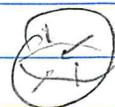
reaction

$\exists$ frame
s.t.

$$R_m = \begin{pmatrix} \lambda & \mu \\ \mu & \nu \end{pmatrix} \Rightarrow Q(R_m) = \begin{pmatrix} \lambda^2 + \mu\nu & \lambda\mu + \mu\nu \\ \lambda\mu + \mu\nu & \mu^2 + 2\nu\mu \\ \mu^2 + 2\nu\mu & \nu^2 + 2\mu\nu \end{pmatrix}$$

$\Rightarrow$ pos. Ricci preserved.

& eigenvalues approach each other.



$g_{ij}(t)$ becomes rounder and shrinks to a point.

To guarantee: either - keep ^{some sort of} positivity and remove ^{restriction} λ in
(Hamilton '86, Bohn-Wilking, Brendle-Schoen
a priori)

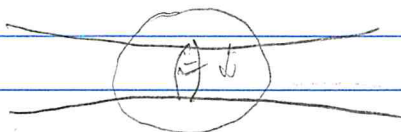
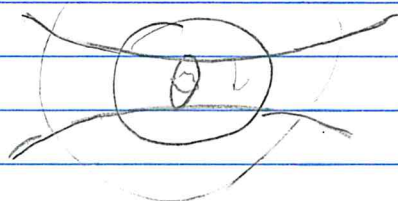
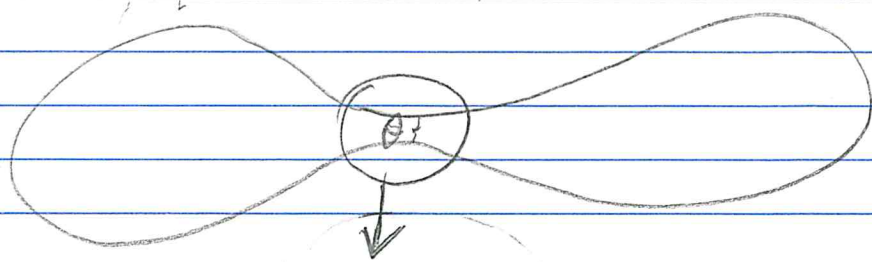
Or: stay in dim 3 w/ curvature assumptions!

Main difficulty: analysis of finite-time singularities. $\left[\text{These do happen.} \right]$

Blowup limit:

has nonnegative curvature

(Hamilton-duey
pinching estimate.)



Conj: (Ham.) Any f.t.

blowup limit is either S^3 or $S^2 \times \mathbb{R}$

(Self-similar goes to Ref.)

why? that's how you get it!

if so, can do surgery:



To bound the curvature AND

undo a connect sum, then restart the flow. Geometrization.

(Singularities fit in with geometric picture.)

Perelman (2002): $\int R dV_g$

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1 Ricci flow is a gradient flow! (mod diffos.)

$$F = - \int (R + |\nabla f|_g) e^{-f} dV_g \rightarrow$$

(D'Nolani: IAS

$$\text{if } (\Delta + \Delta - R) e^{-f} = 0.$$

lecturer on string theory)

[Much stronger f.t. singularity]

F can be localized in general ways. analysis than expected.

→ Proof of Poincaré (and geometrization) conjectures.

50-50.

h p

→ not only b/c I don't know the proof, but b/c it's subtle

Dimension 4: Smooth Poincaré open

Ricci flow presently hopeless:

• Singularities don't necessarily have pinching; curvature

• No plan for surgery. (to bound curvature w.l.o.

changing the info too much.)

(Do something much easier.)

(RG flow for lambda model!)

③ Yang-Mills flow. Fix metric, evolve connection
(much less studied!)

$$\partial_t A = -D_A^* F A$$

$$\partial_t A_i^* \beta = \nabla^L F_{ki}^* \beta$$

convert: on a g.l. E $\downarrow$ $\rightarrow dA + A \wedge A$.

action of $D_A: \mathcal{A}(g_l) \rightarrow \mathcal{A}(g_l)$

$i \rightarrow$ base M
a bundle E

Gradient flow of $Y_M(A) = \frac{1}{2} \int |F_A|^2 dV$ (gradient flow)

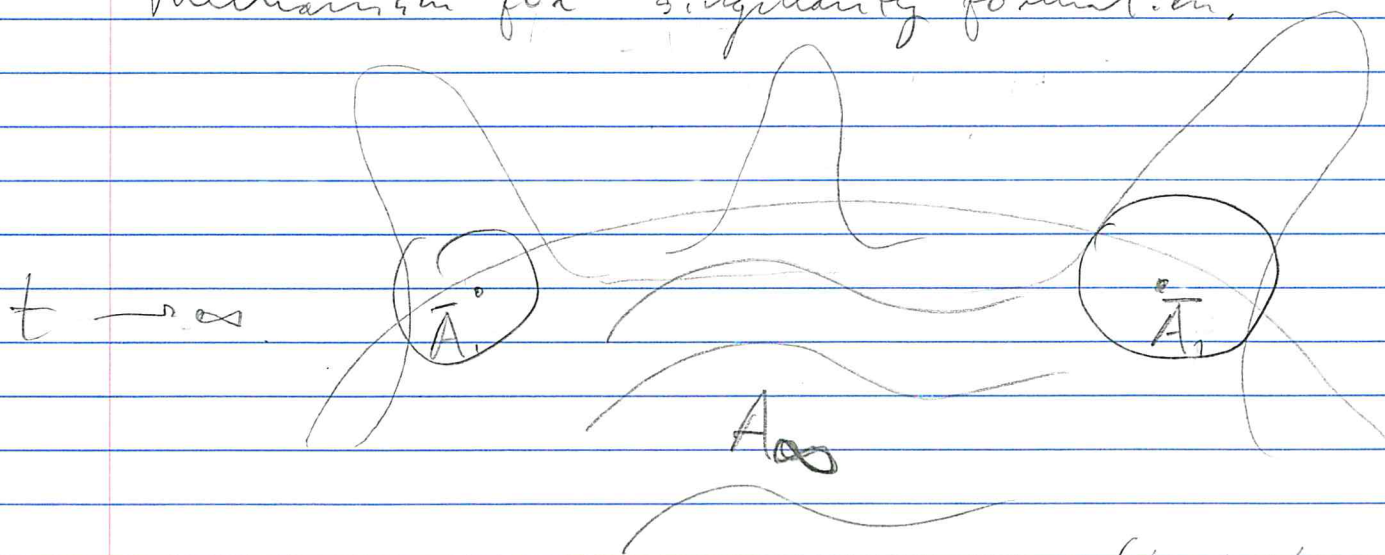
$\Rightarrow$ expect a wk limit $A(t) \rightarrow A_\infty$ which is a crit. pt.

$$D_{A_\infty}^* F_{A_\infty} = 0 \quad (\text{Yang-Mills eqn.})$$

YM is conformally invariant in dim 4!

$$A_\lambda = \lambda A(\lambda x) \Rightarrow Y_M(A_\lambda) = Y_M(A).$$

Mechanism for singularity formation.



Whitcomb compactness: have:

(bundle breaks up,
like dF.)

A_∞ , YM conn on $E_\infty \rightarrow M$, body limit

$\bar{A}_i$, YM conn on $E_i \rightarrow S^4$, blowup limits.
(static soln of flow!)

Donaldson 1985: Assume M is a Kähler surface

Then: • The flow exists for all time. $(\|F_t - A\|_{W.F.}^2)$

• Converges (i.e. $\{A_i\} = \emptyset$)

$(\Rightarrow)$ E is a stable hol'ic bundle.

Kobayashi - Hitchin correspondence.

(Stable $\iff$ flat unitary bundle!)
(Chern class!)

• General Riemannian 4-manifolds?

Recall: f.t. singularities do occur in HM flows!

My thesis: ~~f.t.~~ sing's of $+$ curvature in 4-D YMF
(on $-$)

Bochner f'la + local ~~non-positivity~~ ^{energy inequality}
(like Lichnerowicz-Sampson)

(already diff from HM)

Current work: YMF has no finite-time singularities.
(d.d.)

Conservation laws + decay estimates due to Uhlenbeck

$$\nabla^i S_{ij} \approx 0.$$

(disrupting of blowup limit
from body limit)

sing's are due to
conformal symmetry!

$S_{ij} = \langle F_{ik}, F_{jk} \rangle - \frac{1}{4} g_{ij} |F|^2$ — Stress-energy tensor.

No relevance for Ricci flow, yes for HM flow.